Chapter 4

The Electric Potential

4.1 The Important Stuff

4.1.1 Electrical Potential Energy

A charge q moving in a constant electric field $\mathbf{E}$ experiences a force $\mathbf{F} = q\mathbf{E}$ from that field. Also, as we know from our study of work and energy, the work done on the charge by the field as it moves from point $\mathbf{r}_1$ to $\mathbf{r}_2$ is

$$W = \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{F} \cdot d\mathbf{s}$$

where we mean that we are summing up all the tiny elements of work $dW = \mathbf{F} \cdot d\mathbf{s}$ along the length of the path. When $\mathbf{F}$ is the electrostatic force, the work done is

$$W = \int_{\mathbf{r}_1}^{\mathbf{r}_2} q\mathbf{E} \cdot d\mathbf{s} = q \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{E} \cdot d\mathbf{s} \tag{4.1}$$

In Fig. 4.1, a charge is shown being moved from $\mathbf{r}_1$ to $\mathbf{r}_2$ along two different paths, with $d\mathbf{s}$ and $\mathbf{E}$ shown for a bit of each of the paths.

Now it turns out that from the mathematical form of the electrostatic force, the work done by the force does not depend on the path taken to get from $\mathbf{r}_1$ to $\mathbf{r}_2$. As a result we say

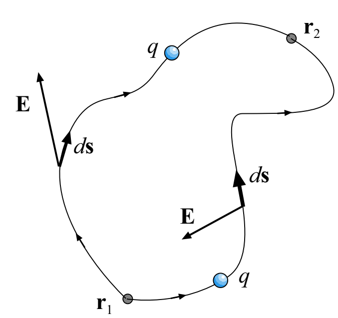

Figure 4.1: Charge is moved from $\mathbf{r}_1$ to $\mathbf{r}_2$ along two separate paths. Work done by the electric force involves the summing up $\mathbf{E} \cdot d\mathbf{s}$ along the path.

that the electric force is **conservative** and it allows us to calculate an **electric potential energy**, which as usual we will denote by U. As before, only the changes in the potential have any real meaning, and the change in potential energy is the negative of the work done by the electric force:

$$\Delta U = -W = -q \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{E} \cdot d\mathbf{s} \tag{4.2}$$

We usually want to discuss the potential energy of a charge *at a particular point*, that is, we would like a function $U(\mathbf{r})$, but for this we need to make a definition for the potential energy *at a particular point*. Usually we will make the choice that the potential energy is zero when the charge is infinitely far away: $U_\infty = 0$.

4.1.2 Electric Potential

Recall how we developed the concept of the electric field $\mathbf{E}$: The force on a charge q_0 is always proportional to q_0, so by dividing the charge out of $\mathbf{F}$ we get something which can conveniently give the force on *any* charge. Likewise, if we divide out the charge q from Eq. 4.2 we get a function which we can use to get the change in potential energy for any charge (simply by multiplying by the charge). This new function is called the **electric potential**, V:

$$\Delta V = \frac{\Delta U}{q}$$

where ΔU is the change in potential energy of a charge q. Then Eq. 4.2 gives us the difference in electrical potential between points $\mathbf{r}_1$ and $\mathbf{r}_2$:

$$\Delta V = - \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{E} \cdot d\mathbf{s} \tag{4.3}$$

The electric potential is a *scalar*. Recalling that it was defined by dividing *potential energy* by *charge* we see that its units are $\frac{\mathrm{J}}{\mathrm{C}}$ (joules per coulomb). The electric potential is of such great importance that we call this combination of units a **volt**[1]. Thus:

$$1\,\text{volt} = 1\,\mathrm{V} = 1\tfrac{\mathrm{J}}{\mathrm{C}} \tag{4.4}$$

Of course, it is then true that a joule is equal to a coulomb-volt=C · V. In general, multiplying a charge times a potential difference gives an *energy*. It often happens that we are multiplying an elementary charge (e) (or some multiple thereof) and a potential difference in volts. It is then convenient to use the unit of energy given by the product of e and a volt; this unit is called the **electron-volt**:

$$1\,\mathrm{eV} = (e) \cdot (1\,\mathrm{V}) = 1.60 \times 10^{-19}\,\mathrm{C} \cdot (1\,\mathrm{V}) = 1.60 \times 10^{-19}\,\mathrm{J} \tag{4.5}$$

Equation 4.3 can only give us the *differences* in the value of the electric potential between two points $\mathbf{r}_1$ and $\mathbf{r}_2$. To arrive at a function $V(\mathbf{r})$ defined at all points we need to specify

[1] Named in honor of the. . . uh. . . French physicist Jim Volt (1813–1743) who did some electrical experiments in. . . um. . . Bologna. That's it, Bologna.

a point at which the potential V is *zero*. Often we will choose this point to be "infinity" (∞) that is, as we get very far away from the set of charges which give the electric field, the potential V becomes very small in absolute value. However this "reference point" can be chosen anywhere and for each problem we need to be sure *where* it is understood that $V = 0$ before we can sensibly talk about the function $V(\mathbf{r})$. Then in Eq. 4.3 equal to this reference point and calculate an potential *function* $V(\mathbf{r})$ for all other points. So we can write:

$$V(\mathbf{r}) = -\int_{\mathbf{r}_{\rm ref}}^{\mathbf{r}} \mathbf{E} \cdot d\mathbf{s} \tag{4.6}$$

4.1.3 Equipotential Surfaces

For a given configuration of charges, a set of points where the electric potential $V(\mathbf{r})$ has a given value is called an **equipotential surface**. It takes no work to move a charged particle from one point on such a surface to another point on the surface, for then we have $\Delta V = 0$.

From the relations between $\mathbf{E}(\mathbf{r})$ and $V(\mathbf{r})$ it follows that the field lines are perpendicular to the equipotential surfaces everywhere.

4.1.4 Finding E from V

The definition of V an *integral* involving the $\mathbf{E}$ field implies that the electric field comes from V by taking *derivatives*:

$$E_x = -\frac{\partial V}{\partial x} \qquad E_y = -\frac{\partial V}{\partial y} \qquad E_z = -\frac{\partial V}{\partial z} \tag{4.7}$$

These relations can be written as one equation using the notation for the gradient:

$$\mathbf{E} = -\nabla V \tag{4.8}$$

4.1.5 Potential of a Point Charge and Groups of Points Charges

Using Eq. 4.3, one can show that if we specify that the electrical potential is zero at "infinity", then the potential due to a point charge q is

$$V(\mathbf{r}) = k\frac{q}{r} = \frac{1}{4\pi\epsilon_0}\frac{q}{r} \tag{4.9}$$

where r is the distance of the charge from the point of interest. Furthermore, for a set of point charges q_1, q_2, q_3, ... the electrical potential is

$$V(\mathbf{r}) = \sum_i k\frac{q_i}{r_i} = \sum_i \frac{1}{4\pi\epsilon_0}\frac{q_i}{r_i} \tag{4.10}$$

where r_i is the distance of each charge from the point of interest.

Using Eq. 4.10, one can show that the potential due to an electric dipole with magnitude p at the origin (pointing upward along the z axis) is

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{p\cos\theta}{r^2} \tag{4.11}$$

Here, r and θ have the usual meaning in spherical coordinates.

4.1.6 Potential Due to a Continuous Charge Distribution

To get the electrical potential due to a continuous distribution of charge (with $V = 0$ at infinity assumed), add up the contributions to the potential; the potential due to a charge dq at distance r is $dV = \frac{1}{4\pi\epsilon_0}\frac{dq}{r}$ so that we must do the integral

$$V = \frac{1}{4\pi\epsilon_0}\int \frac{dq}{r} = \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho(\mathbf{r})d\tau}{r} \tag{4.12}$$

In the last expression we are using the charge density $\rho(\mathbf{r})$ of the distribution to get the element of charge dq for the volume element $d\tau$.

4.1.7 Potential Energy of a System of Charges

The potential energy of a pair of point charges (i.e. the work W needed to bring point charges q_1 and q_2 from infinite separation to a separation r) is

$$U = W = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r} \tag{4.13}$$

For a larger set of charges the potential energy is given by the sum

$$U = U_{12} + U_{23} + U_{13} + \ldots = \frac{1}{4\pi\epsilon_0}\sum_{\text{pairs } ij}\frac{q_i q_j}{r_{ij}} \tag{4.14}$$

Here r_{ij} is the distance between charges q_i and q_j. Each pair is only counted *once* in the sum.

4.2 Worked Examples

4.2.1 Electric Potential

1. The electric potential difference between the ground and a cloud in a particular thunderstorm is 1.2×10^9 V. What is the magnitude of the change in energy (in multiples of the electron-volt) of an electron that moves between the ground and the cloud?

The *magnitude* of the change in potential as the electron moves between ground and cloud (we don't care which way) is $|\Delta V| = 1.2 \times 10^9\,\mathrm{V}$. Multiplying by the magnitude of the electron's charge gives the *magnitude* of the change in potential energy. Note that lumping "e" and "V" together gives the eV (electron-volt), a unit of energy:

$$|\Delta U| = |q\Delta V| = e(1.2 \times 10^9\,\mathrm{V}) = 1.2 \times 10^9\,\mathrm{eV} = 1.2\,\mathrm{GeV}$$

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2. An infinite nonconducting sheet has a surface charge density $\sigma = 0.10\mu\mathrm{C/m}^2$ on one side. How far apart are equipotential surfaces whose potentials differ by $50\,\mathrm{V}$?

In Chapter 3, we encountered the formula for the electric field due a nonconducting sheet of charge. From Eq. 3.5, we had: $E_z = \sigma/(2\epsilon_0)$, where σ is the charge density of the sheet, which lies in the xy plane. So the plane of charge in this problem gives rise to an E field:

$$\begin{aligned} E_z &= \frac{\sigma}{2\epsilon_0} \\ &= \frac{(0.10 \times 10^{-6}\,\frac{\mathrm{C}}{\mathrm{m}^2}}{2(8.85 \times 10^{-12}\,\frac{\mathrm{C}^2}{\mathrm{N\cdot m^2}})} = 5.64 \times 10^3\,\frac{\mathrm{N}}{\mathrm{C}} \end{aligned}$$

Here the E field is *uniform* and also $E_x = E_y = 0$.

Now, from Eq. 4.7 we have

$$\frac{\partial V}{\partial z} = -E_z = -5.64 \times 10^3\,\frac{\mathrm{N}}{\mathrm{C}}\ .$$

and when the rate of change of some quantity (in this case, with respect to the z coordinate) is constant we can write the relation in terms of *finite* changes, that is, with "Δ"s:

$$\frac{\Delta V}{\Delta z} = -E_z = -5.64 \times 10^3\,\frac{\mathrm{N}}{\mathrm{C}}$$

and from this result we can find the change in z corresponding to any change in V. If we are interested in $\Delta V = 50\,\mathrm{V}$, then

$$\Delta z = -\frac{\Delta V}{E_z} = -\frac{(50\,\mathrm{V})}{(5.64 \times 10^3\,\frac{\mathrm{N}}{\mathrm{C}})} = -8.8 \times 10^{-3}\,\mathrm{m} = -8.8\,\mathrm{mm}$$

i.e. to get a change in potential of $+50\,\mathrm{V}$ we need a change in z coordinate of $-8.8\,\mathrm{mm}$.

Since the potential only depends on the distance from the plane, the equipotential surfaces are *planes*. The distance between planes whose potential differs by $50\,\mathrm{V}$ is $8.8\,\mathrm{mm}$.

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3. Two large, parallel conducting plates are $12\,\mathrm{cm}$ apart and have charges of equal magnitude and opposite sign on their facing surfaces. An electrostatic force of $3.9 \times 10^{-15}\,\mathrm{N}$ acts on an electron placed anywhere between the two plates.

(Neglect fringing.) (a) Find the electric field at the position of the electron. (b) What is the potential difference between the plates?

(a) We are given the *magnitude* of the electric force on an electron (whose charge is $-e$). Then the magnitude of the E field must be:

$$E = \frac{F}{|q|} = \frac{F}{e} = \frac{(3.9 \times 10^{-15}\,\mathrm{N})}{(1.60 \times 10^{-19}\,\mathrm{C})} = 2.4 \times 10^4\,\frac{\mathrm{N}}{\mathrm{C}} = 2.4 \times 10^4\,\frac{\mathrm{V}}{\mathrm{m}}$$

(b) The $\mathbf{E}$ field in the region between two large oppositely–charged plates is *uniform* so in that case, we can write

$$E_x = -\frac{\Delta V}{\Delta x}$$

(where the $\mathbf{E}$ field points in the x direction, i.e. perpendicular to the plates), and the potential difference between the plates has magnitude

$$|\Delta V| = |E_x \Delta x| = (2.4 \times 10^4\,\tfrac{\mathrm{V}}{\mathrm{m}})(0.12\,\mathrm{m}) = 2.9 \times 10^3\,\mathrm{V}$$

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4. The electric field inside a nonconducting sphere of radius R with charge spread uniformly throughout its volume, is radially directed and has magnitude

$$E(r) = \frac{qr}{4\pi\epsilon_0 R^3} \quad .$$

Here q (positive or negative) is the total charge within the sphere, and r is the distance from the sphere's center. (a) Taking $V = 0$ at the center of the sphere, find the electric potential $V(r)$ inside the sphere. (b) What is the difference in electric potential between a point on the surface and the sphere's center? (c) If q is positive, which of those two points is at the higher potential?

(a) We will use Eq. 4.6 to calculate $V(r)$ using $r = 0$ as the reference point: $V(0) = 0$. The electric field has only a radial component $E_r(r)$ so that we will evaluate:

$$V(\mathbf{r}) = -\int_{\mathbf{r}_{\mathrm{ref}}}^{\mathbf{r}} \mathbf{E} \cdot d\mathbf{s} = -\int_0^r E_r(r')dr'$$

Using the given expression for $E_r(r')$ (which one can *derive* using Gauss'(s) law) we get:

$$\begin{aligned} V(r) &= -\int_0^r \frac{qr'}{4\pi\epsilon_0 R^3}dr = -\frac{q}{4\pi\epsilon_0 R^3}\int_0^r r'\,dr' \\ &= -\frac{q}{4\pi\epsilon_0 R^3}\frac{r^2}{2} \\ &= -\frac{qr^2}{8\pi\epsilon_0 R^3} \end{aligned}$$

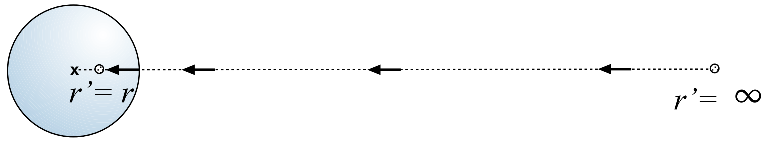

Figure 4.2: Path of integration for Example 5. Integration goes from $r' = \infty$ to $r' = r$.

(b) Using the result of part (a), the difference between values of $V(r)$ on the sphere's surface and at its center is

$$V(R) - V(0) = -\frac{qR^2}{8\pi\epsilon_0 R^3} = -\frac{q}{8\pi\epsilon_0 R}$$

(c) For q positive, the answer to part (b) is a *negative* number, so the center of the sphere must be at a higher potential.

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5. A charge q is distributed uniformly throughout a spherical volume of radius R. (a) Setting $V = 0$ at infinity, show that the potential at a distance r from the center, where $r < R$, is given by

$$V = \frac{q(3R^2 - r^2)}{8\pi\epsilon_0 R^3} \ .$$

(b) Why does this result differ from that of the previous example? (c) What is the potential difference between a point of the surface and the sphere's center? (d) Why doesn't this result differ from that of the previous example?

(a) We find the function $V(r)$ just as we did the last example, but this time the reference point (the place where $V = 0$) is at $r = \infty$. So we will evaluate:

$$V(\mathbf{r}) = -\int_{\mathbf{r}_{\mathrm{ref}}}^{\mathbf{r}} \mathbf{E} \cdot d\mathbf{s} = -\int_{\infty}^{r} E_r(r')\, dr' \ . \tag{4.15}$$

The integration path is shown in Fig. 4.2. We note that the integration (from $r' = \infty$ to $r' = r$ with $r < R$) is over values of r both outside and inside the sphere.

Just as before, the E field for points *inside* the sphere is

$$E_{r,\ \mathrm{in}}(r) = \frac{qr}{4\pi\epsilon_0 R^3} \ , \tag{4.16}$$

but now we will also need the value of the E field *outside* the sphere. By Gauss'(s) law the external E field is that same as that due to a point charge q at distance r, so:

$$E_{r,\ \mathrm{out}}(r) = \frac{q}{4\pi\epsilon_0 r^2} \ . \tag{4.17}$$

Because $E_r(r)$ has two different forms for the interior and exterior of the sphere, we will have to split up the integral in Eq. 4.15 into two parts. When we go from ∞ to R we need to use Eq. 4.17 for $E_r(r')$. When we go from R to r we need to use Eq. 4.16 for $E_r(r')$. So from Eq. 4.15 we now have

$$\begin{aligned} V(r) &= -\int_{\infty}^{R} E_{r,\,\mathrm{out}}(r')\,dr' - \int_{R}^{r} E_{r,\,\mathrm{in}}(r')\,dr' \\ &= -\int_{\infty}^{R} \left(\frac{q}{4\pi\epsilon_0 r'^2}\right) dr' - \int_{R}^{r} \left(\frac{qr'}{4\pi\epsilon_0 R^3}\right) dr' \\ &= -\frac{q}{4\pi\epsilon_0}\left\{\int_{\infty}^{R} \frac{dr'}{r'^2} + \int_{R}^{r} \frac{r'}{R^3}\,dr'\right\} \end{aligned}$$

Now do the individual integrals and we're done:

$$\begin{aligned} V(r) &= -\frac{q}{4\pi\epsilon_0}\left\{-\frac{1}{r'}\bigg|_{\infty}^{R} + \frac{r'^2}{2R^3}\bigg|_{R}^{r}\right\} \\ &= -\frac{q}{4\pi\epsilon_0}\left\{-\frac{1}{R} + \frac{(r^2-R^2)}{2R^3}\right\} \\ &= \frac{q}{4\pi\epsilon_0}\left(\frac{2R^2}{2R^3} + \frac{(R^2-r^2)}{2R^3}\right) \\ &= \frac{q(3R^2-r^2)}{8\pi\epsilon_0 R^3} \end{aligned}$$

(b) The difference between this result and that of the previous example is due to the different choice of reference point. There is no problem here since it is only the *differences* in electrical potential that have any meaning in physics.

(c) using the result of part (a), we calculate:

$$\begin{aligned} V(R) - V(0) &= \frac{q(2R^2)}{8\pi\epsilon_0 R^3} - \frac{q(3R^2)}{8\pi\epsilon_0 R^3} \\ &= -\frac{qR^2}{8\pi\epsilon_0 R^3} = -\frac{q}{8\pi\epsilon_0 R} \end{aligned}$$

This is the *same* as the corresponding result in the previous example.

(d) *Differences* in the electrical potential will *not* depend on the choice of the reference point, the answer *should* be the same as in the previous example... if $V(r)$ is calculated correctly!

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6. What are (a) the charge and (b) the charge density on the surface of a conducting sphere of radius $0.15\,\mathrm{m}$ whose potential is $200\,\mathrm{V}$ (with $V = 0$ at infinity)?

(a) We are given the radius R of the conducting sphere; we are asked to find its charge Q.

From our work with Gauss'(s) law we know that the electric field outside the sphere is the same as that of a point charge Q at the sphere's center. Then if we were to use Eq. 4.6

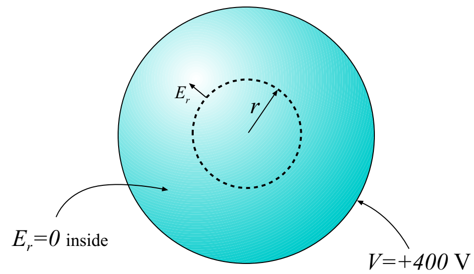

Figure 4.3: Conducting charged sphere, has potential of 400 V, from Example 7.

with the condition $V = 0$ at infinity (which *is* outside the sphere!), we would get the *same* result for V as we would for a point charge Q at the origin and $V = 0$ at infinity, namely:

$$V(r) = \frac{1}{4\pi\epsilon_0}\frac{Q}{r} \qquad \text{(outside sphere)}$$

This equation holds for $r \geq R$.

Then at the sphere's surface ($r = R$) we have:

$$V = \frac{1}{4\pi\epsilon_0}\frac{Q}{R}$$

Solve for Q and plug in the numbers:

$$\begin{aligned} Q &= 4\pi\epsilon_0 VR \\ &= 4\pi(8.85 \times 10^{-12}\,\tfrac{\mathrm{C}^2}{\mathrm{N\cdot m}^2})(200\,\mathrm{V})(0.15\,\mathrm{m}) \\ &= 3.3 \times 10^{-9}\,\mathrm{C} \end{aligned}$$

The charge on the sphere is 3.3×10^{-9} C.

(b) The charge found in (a) resides on the surface of the conducting sphere. To get the charge density, divide the charge by the surface are of the sphere:

$$\sigma = \frac{Q}{4\pi R^2} = \frac{(3.3 \times 10^{-9}\,\mathrm{C})}{4\pi(0.15\,\mathrm{m})2} = 1.2 \times 10^{-8}\,\frac{\mathrm{C}}{\mathrm{m}^2}$$

The charge density on the sphere's surface is $1.2 \times 10^{-8}\,\frac{\mathrm{C}}{\mathrm{m}^2}$.

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7. An empty hollow metal sphere has a potential of $+400\,\mathrm{V}$ with respect to ground (defined to be at $V = 0$) and has a charge of $5.0 \times 10^9\,\mathrm{C}$. Find the electric potential at the center of the sphere.

The problem is diagrammed in Fig. 4.3. From considering a spherical Gaussian surface drawn inside the sphere, we see that the electric field E_r must be *zero* everywhere in side the sphere because such a surface will enclose no charge. But for spherical geometries, E_r and V are related by

$$E_r = -\frac{dV}{dr}$$

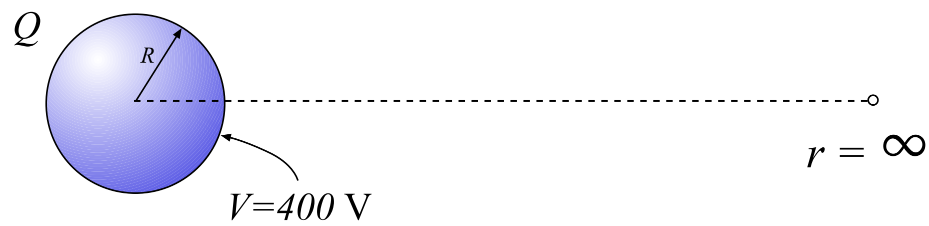

Figure 4.4: Conducting charged sphere, has potential of 400 V (with $V = 0$ at $r = \infty$), from Example 8.

so that with $E_r = 0$, V must be constant throughout the interior of the spherical conductor. Since the value of V on the sphere itself is $+400\,\mathrm{V}$, V then must also equal $+400\,\mathrm{V}$ at the center.

So $V = +400\,\mathrm{V}$ at the center of the sphere. (There was no calculating to do on this problem!)

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8. What is the excess charge on a conducting sphere of radius $R = 0.15\,\mathrm{m}$ if the potential of the sphere is $1500\,\mathrm{V}$ and $V = 0$ at infinity?

The problem is diagrammed in Fig. 4.4. If the sphere has net charge Q then from Gauss' law the radial component of the electric field for points outside the sphere is

$$E_r = k\frac{Q}{r^2}$$

Using Eq. 4.6 with $r = \infty$ as the reference point, the potential at distance R from the sphere's center is:

$$\begin{aligned} V(R) &= -\int_\infty^r E_r dr = -\int_\infty^r \frac{kQ}{r^2} dr \\ &= \frac{kQ}{r}\bigg|_\infty^R = \frac{kQ}{R} - 0 \\ &= \frac{kQ}{R} \end{aligned}$$

(Note that the integration takes place over values of r *outside* the sphere so that the expression for E_r is the correct one. E_r is zero *inside* the sphere.)

We are given that $V(R) = 400\,\mathrm{V}$, so from $kQ/R = 400\,\mathrm{V}$ we solve for Q and get:

$$Q = \frac{R(400\,\mathrm{V})}{k} = \frac{(0.15\,\mathrm{m})(400\,\mathrm{V})}{(8.99\times 10^9\,\frac{\mathrm{N\cdot m^2}}{\mathrm{C^2}})} = 2.5\times 10^{-8}\,\mathrm{C}$$

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9. The electric potential at points in an xy plane is given by

$$V = (2.0\,\tfrac{\mathrm{V}}{\mathrm{m^2}})x^2 - (3.0\,\tfrac{\mathrm{V}}{\mathrm{m^2}})y^2\ .$$

What are the magnitude and direction of the electric field at the point $(3.0\,\mathrm{m}, 2.0\,\mathrm{m})$**?**

Equations 4.7 show how to get the components of the **E** field if we have the electric potential V as a function of x and y. Taking partial derivatives, we find:

$$E_x = -\frac{\partial V}{\partial x} = -(4.0\,\tfrac{\mathrm{V}}{\mathrm{m}^2})x \qquad \text{and} \qquad E_y = -\frac{\partial V}{\partial y} = +(6.0\,\tfrac{\mathrm{V}}{\mathrm{m}^2})y \;\; .$$

Plugging in the given values of $x = 3.0\,\mathrm{m}$ and $y = 2.0\,\mathrm{m}$ we get:

$$E_x = -12\,\tfrac{\mathrm{V}}{\mathrm{m}} \qquad \text{and} \qquad E_y = -12\,\tfrac{\mathrm{V}}{\mathrm{m}}$$

So the magnitude of the **E** field at the given is

$$E = \sqrt{(12.0)^2 + (12.0)^2}\,\tfrac{\mathrm{V}}{\mathrm{m}} = 17\,\tfrac{\mathrm{V}}{\mathrm{m}}$$

and its direction is given by

$$\theta = \tan^{-1}\left(\frac{E_y}{E_x}\right) = \tan^{-1}(1.0) = 135^\circ$$

where for θ we have made the proper choice so that it lies in the second quadrant.

4.2.2 Potential Energy of a System of Charges

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10. (a) What is the electric potential energy of two electrons separated by $2.00\,\mathrm{nm}$**? (b) If the separation increases, does the potential energy increase or decrease?**

Since the charge of an electron is $-e$, using Eq. 4.13 we find:

$$\begin{aligned} U &= \frac{1}{4\pi\epsilon_0}\frac{(-e)(-e)}{r} \\ &= \frac{1}{4\pi(8.85\times 10^{-12}\,\frac{\mathrm{C}^2}{\mathrm{N\cdot m^2}})}\frac{(1.60\times 10^{-19}\,\mathrm{C})^2}{(2.00\times 10^{-9}\,\mathrm{m})} \\ &= 1.15\times 10^{-19}\,\mathrm{J} \end{aligned}$$

As the charges are both positive, the potential energy is a positive number and is inversely proportional to r. So the potential energy *decreases* as r increases.

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11. Derive an expression for the work required to set up the four-charge configuration of Fig. 4.5, assuming the charges are initially infinitely far apart.

The work required to set up these charges is the same as the potential energy of a set of point charges, given in Eq. 4.14. (That is, sum the potential energies $k\frac{q_iq_j}{r_{ij}}$ over all pairs of

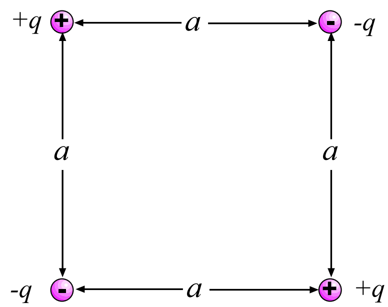

Figure 4.5: Charge configuration for Example 11.

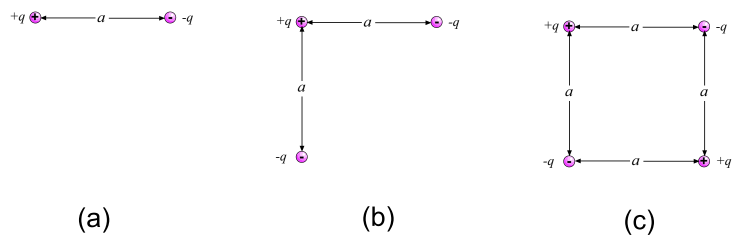

Figure 4.6: (a) Second charge is brought in from ∞ and put in place. (b) Third charge is brought in. (c) Last charge is brought in.

charges.) We can arrive at the same answer and understand that formula a little better if we assemble the system one charge at a time.

Begin with the charge in the upper left corner of Fig. 4.5. Moving this charge from infinity to the desired location requires *no* work because it is never near any other charge. We can write: $W_1 = 0$.

Now bring up the charge in the upper right corner ($-q$). Now we have the configuration shown in Fig. 4.6(a). While being put into place it has experienced a force from the first charge and the work required of the external agency is the change in potential energy of this charge, namely

$$W_2 = \frac{1}{4\pi\epsilon_0}\frac{(+q)(-q)}{a} = -\frac{q^2}{4\pi\epsilon_0 a}$$

Now bring the charge in the lower left corner ($-q$), as shown in 4.6(b). When put into place it is a distance a from the first charge and $\sqrt{2}a$ from the second charge. The work required for this step is the potential energy of the third charge in this configuration, namely:

$$W_3 = \frac{1}{4\pi\epsilon_0}\frac{(+q)(-q)}{a} + \frac{1}{4\pi\epsilon_0}\frac{(-q)(-q)}{\sqrt{2}a} = \frac{q^2}{4\pi\epsilon_0 a}\left(-1 + \frac{1}{\sqrt{2}}\right)$$

Finally, bring in the fourth charge ($+q$) to give the configuration in Fig. 4.6(c). The last charge is now a distance a from *two* $-q$ charges and a distance $\sqrt{2}a$ from the other $+q$ charge. So the work required for this step is

$$W_4 = 2\frac{1}{4\pi\epsilon_0}\frac{(+q)(-q)}{a} + \frac{1}{4\pi\epsilon_0}\frac{(+q)(+q)}{\sqrt{2}a}$$

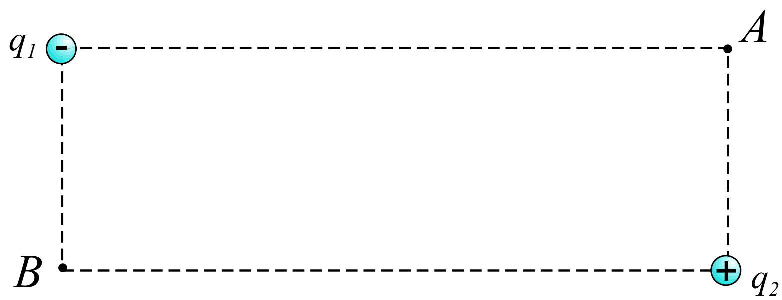

Figure 4.7: Charge configuration for Example 12.

$$= \frac{q^2}{4\pi\epsilon_0 a}\left(-2 + \frac{1}{\sqrt{2}}\right)$$

So now add up all the W's to get the total work done:

$$\begin{aligned} W_{\rm Total} &= W_1 + W_2 + W_3 + W_4 \\ &= \frac{q^2}{4\pi\epsilon_0 a}\left(-1 - 1 + \frac{1}{\sqrt{2}} - 2 + \frac{1}{\sqrt{2}}\right) \\ &= \frac{q^2}{4\pi\epsilon_0 a}\left(-4 + \frac{2}{\sqrt{2}}\right) \end{aligned}$$

This is a nice analytic answer; if we combine all the numerical factors (including the 4π) we get:

$$W_{\rm Total} = \frac{(-0.21)q^2}{\epsilon_0 a}$$

This is the same result as we'd get by using Eq. 4.14.

---

12. In the rectangle of Fig. 4.7, the sides have lengths $5.0\,\text{cm}$ and $15\,\text{cm}$, $q_1 = -5.0\,\mu\text{C}$ and $q_2 = +2.0\,\mu\text{C}$. With $V = 0$ at infinity, what are the electric potentials (a) at corner A and (b) corner B? (c) How much work is required to move a third charge $q_3 = +3.0\,\mu\text{C}$ from B to A along a diagonal of the rectangle? (d) Does this work increase or decrease the electric energy of the three–charge system? Is more, less or the same work required if q_3 is moved along paths that are (e) inside the rectangle but not on the diagonal and (f) outside the rectangle?

(a) To find the electric potential due to a group of point charges, use Eq. 4.10. Since point A is 15 cm away from the $-5.0\,\mu\text{C}$ charge and 5.0 cm away from the $+2.0\,\mu\text{C}$ charge, we get:

$$\begin{aligned} V &= \frac{1}{4\pi\epsilon_0}\left[\frac{q_1}{r_1} + \frac{q_2}{r_2}\right] \\ &= (8.99\times 10^9\,\tfrac{\text{N}\cdot\text{m}^2}{\text{C}^2})\left[\frac{(-5.0\times 10^{-6}\,\text{C})}{(15\times 10^{-2}\,\text{m})} + \frac{(+2.0\times 10^{-6}\,\text{C})}{(5.0\times 10^{-2}\,\text{m})}\right] = 6.0\times 10^4\,\text{V} \end{aligned}$$

(b) Perform the same calculation as in part (a). The charges q_1 and q_2 are at different distances from point B so we get a different answer:

$$V = (8.99 \times 10^9 \, \tfrac{\mathrm{N \cdot m^2}}{\mathrm{C^2}}) \left[\frac{(-5.0 \times 10^{-6}\,\mathrm{C})}{(5.0 \times 10^{-2}\,\mathrm{m})} + \frac{(+2.0 \times 10^{-6}\,\mathrm{C})}{(15 \times 10^{-2}\,\mathrm{m})} \right] = -7.8 \times 10^5\,\mathrm{V}$$

(c) Using the results of part (a) and (b), calculate the change in potential V as we move from point B to point A:

$$\Delta V = V_A - V_B = 6.0 \times 10^4\,\mathrm{V} - (-7.8 \times 10^5\,\mathrm{V}) = 8.4 \times 10^5\,\mathrm{V}$$

The change in potential energy for a $+3.0\,\mu\mathrm{C}$ charge to move from B to A is

$$\Delta U = q\Delta V = (3.0 \times 10^{-6}\,\mathrm{C})(8.4 \times 10^5\,\mathrm{V}) = 2.5\,\mathrm{J}$$

(d) Since a positive amount of work is done by the outside agency in moving the charge from B to A, the electric energy of the system has *increased*. We can see that this must be the case because the $+3.0\,\mu\mathrm{C}$ charge has been moved closer to another positive charge and farther away from a negative charge.

(e) The force which a point charge (or set of point charges) exerts on a another charge is a *conservative* force. So the work which it does (or likewise the work *required* of some outside force) as the charge moves from one point to another is *independent of the path taken*. Therefore we would require the same amount of work if the path taken was some other path inside the rectangle.

(f) Since the work done is independent of the path taken, we require the same amount of work even if the path from A to B goes outside the rectangle.

---

13. Two tiny metal spheres A and B of mass $m_A = 5.00\,\mathrm{g}$ and $m_B = 10.0\,\mathrm{g}$ have equal positive charges $q = 5.00\,\mu\mathrm{C}$. The spheres are connected by a massless nonconducting string of length $d = 1.00\,\mathrm{m}$, which is much greater than the radii of the spheres. (a) What is the electric potential energy of the system? (b) Suppose you cut the string. At that instant, what is the acceleration of each sphere? (c) A long time after you cut the string, what is the speed of each sphere?

(a) The initial configuration of the charges in shown in Fig. 4.8(a). The electrostatic potential energy of this system (i.e. the work needed to bring the charges together from far away is

$$U = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r} = (8.99 \times 10^9 \, \tfrac{\mathrm{N \cdot m^2}}{\mathrm{C^2}}) \frac{(5.00 \times 10^{-6}\,\mathrm{C})^2}{(1.00\,\mathrm{m})} = 0.225\,\mathrm{J}$$

We are justified in using formulae for *point charges* because the problem states that the sizes of the spheres are small compared to the length of the string (1.00 m).

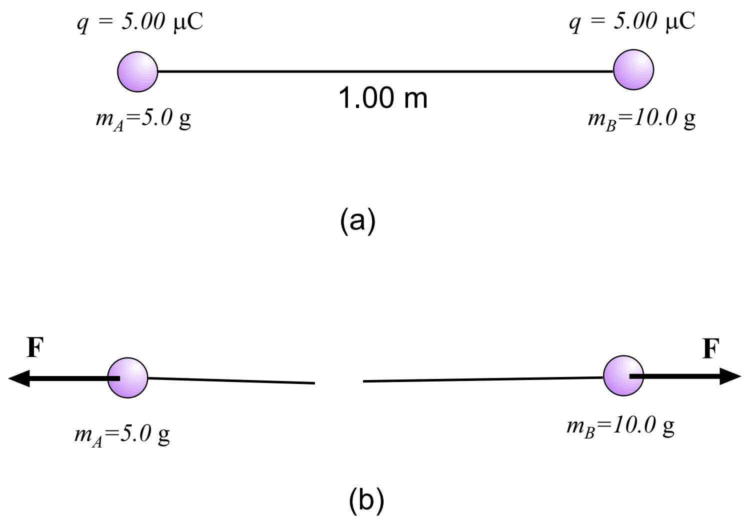

Figure 4.8: (a) Charged spheres attached to a string, in Example 13. The electrostatic repulsion is balanced by the string tension. (b) After string is cut there is a mutual force of electrical repulsion **F**. Magnitude of the *force* on each charge is the same but their *accelerations* are different!

(b) From Coulomb's law, the magnitude of the mutual force of repulsion of the two charges is

$$F = \frac{1}{4\pi\epsilon_0}\frac{q^2}{r^2} = (8.99\times 10^9\,\tfrac{\mathrm{N\cdot m^2}}{\mathrm{C^2}})\frac{(5.00\times 10^{-6}\,\mathrm{C})^2}{(1.00\,\mathrm{m})^2} = 0.225\,\mathrm{N}$$

but since the masses of the spheres are different their *accelerations* have different magnitudes. From Newton's 2$^{\mathrm{nd}}$ law, the accelerations of the masses are:

$$a_1 = \frac{F}{m_1} = \frac{(0.225\,\mathrm{N})}{(5.00\times 10^{-3}\,\mathrm{kg})} = 45.0\,\tfrac{\mathrm{m}}{\mathrm{s^2}}$$

$$a_2 = \frac{F}{m_2} = \frac{(0.225\,\mathrm{N})}{(10.0\times 10^{-3}\,\mathrm{kg})} = 22.2\,\tfrac{\mathrm{m}}{\mathrm{s^2}}$$

Of course, the accelerations are in opposite *directions*.

(c) From the time that the string breaks to the time that we can say that the spheres are "very far apart", the only force that each one experiences is the force of electrical repulsion (arising from the other sphere). This is a *conservative force* so that total mechanical energy is conserved. It is also true that there are no *external forces* being exerted on the two–sphere system. Then we know that the total (vector) momentum of the system is also conserved.

First, let's deal with the condition of energy conservation. The total energy right after the string is cut is just the potential energy found in part (a) since the spheres are not yet in motion. So $E_{\mathrm{init}} = 0.225\,\mathrm{J}$.

When the spheres are a long ways apart, there is no electrical potential energy, but they are in motion with respective speeds v_A and v_B so there is kinetic energy at "large" separation. Then energy conservation tells us:

$$\tfrac{1}{2}m_A v_A^2 + \tfrac{1}{2}m_B v_B^2 = 0.225\,\mathrm{J} \tag{4.18}$$

Momentum conservation gives us the other equation that we need. If mass B has x–velocity v_B then mass A has x–velocity $-v_A$ (it moves in the other direction. The system begins and ends with a total momentum of *zero* so then:

$$-m_A v_A + m_B v_B = 0 \qquad \Longrightarrow \qquad v_B = \frac{m_A}{m_B} v_A$$

Substitute this result into 4.18 and get:

$$\tfrac{1}{2} m_A v_A^2 + \tfrac{1}{2} m_B \left(\frac{m_A^2}{m_B^2} \right) v_A^2 = 0.225\,\mathrm{J}$$

Factor out v_A^2 on the left side and plug in some numbers:

$$\tfrac{1}{2} \left(m_A + \frac{m_A^2}{m_B} \right) v_A^2 = \tfrac{1}{2} \left(5.00\,\mathrm{g} + \frac{(5.00\,\mathrm{g})^2}{(10.0\,\mathrm{g})} \right) v_A^2 = (3.75 \times 10^{-3}\,\mathrm{kg}) v_A^2 = 0.225\,\mathrm{J}$$

So then we get the final speed of A:

$$v_A^2 = \frac{0.225\,\mathrm{J}}{3.75 \times 10^{-3}\,\mathrm{kg}} = 60.0\,\tfrac{\mathrm{m}^2}{\mathrm{s}^2} \qquad \Longrightarrow v_A = 7.75\,\tfrac{\mathrm{m}}{\mathrm{s}}$$

and the speed of B:

$$v_B = \frac{m_A}{m_B} v_A = \left(\frac{5.00\,\mathrm{g}}{10.0\,\mathrm{g}} \right) 7.75\,\tfrac{\mathrm{m}}{\mathrm{s}} = 3.87\,\tfrac{\mathrm{m}}{\mathrm{s}}$$

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14. Two electrons are fixed 2.0 cm apart. Another electron is shot from infinity and stops midway between the two. What is its initial speed?

The problem is diagrammed in Fig. 4.9(a) and (b). Since the electrostatic force is a conservative force, we know that *energy is conserved* between configurations (a) and (b). In picture (a) there is energy stored in the repulsion of the pair of electrons as well as the kinetic energy of the third electron. (Initially the third electron is too far away to "feel" the first two electrons.) In picture (b) there is no kinetic energy but the electrical potential energy has increased due to the repulsion between the third electron and the first two. If we can calculate the change in potential energy ΔU then by using energy conservation, $\Delta U + \Delta K = 0$ we can find the initial speed of the electron.

The potential energy of a set of point charges (with $V = 0$ at ∞) is given in Eq. 4.14. When the third electron comes from infinity and stops at the midpoint, the increase in potential energy the contribution given by the third electron as it "sees" its new neighbors. With $r = 1.0\,\mathrm{cm}$, this increase is

$$\Delta U = \frac{1}{4\pi\epsilon_0} \frac{(-e)(-e)}{r} + \frac{1}{4\pi\epsilon_0} \frac{(-e)(-e)}{r} = \frac{e^2}{2\pi\epsilon_0 r}$$

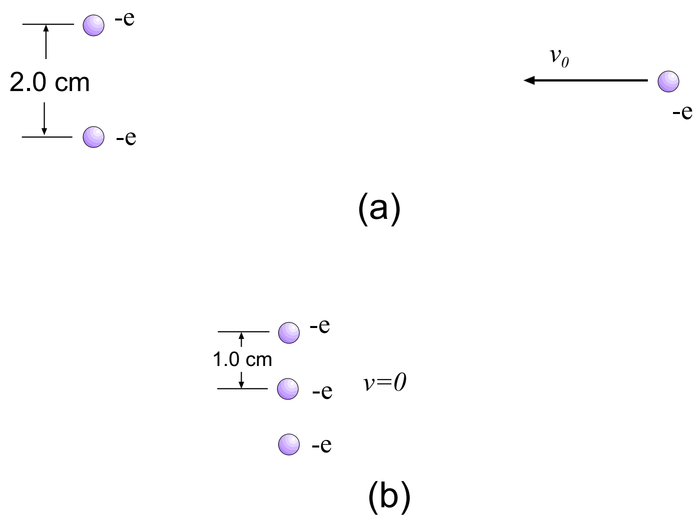

Figure 4.9: (a) Electron flies in from ∞ with speed v_0. (b) Electron comes to rest midway between the other two electrons.

The change in kinetic energy is $\Delta K = -\frac{1}{2}mv_0^2$. Then energy conservation gives:

$$\Delta K = -\Delta U \qquad \Longrightarrow \qquad -\tfrac{1}{2}mv_0^2 = -\frac{e^2}{2\pi\epsilon_0 r}$$

Solve for v_0:

$$\begin{aligned} v_0^2 &= \frac{e^2}{\pi\epsilon_0 m r} \\ &= \frac{(1.60 \times 10^{-19}\,\mathrm{C})^2}{\pi(8.85 \times 10^{-12}\,\frac{\mathrm{C}^2}{\mathrm{N\cdot m^2}})(9.11 \times 10^{-31}\,\mathrm{kg})(1.0 \times 10^{-2}\,\mathrm{m})} = 1.01 \times 10^5\,\frac{\mathrm{m}^2}{\mathrm{s}^2} \end{aligned}$$

which gives

$$v_0 = 3.18 \times 10^2\,\frac{\mathrm{m}}{\mathrm{s}}$$